Hamiltonian thermodynamics of charged black holes

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We consider the most general diffeomorphism invariant action in 1+1 spacetime dimensions that contains a metric, dilaton and Abelian gauge field, and has at most second derivatives of the fields. Our action contains a topological term (linear in the Abelian field strength) that has not been considered in previous work. We impose boundary conditions appropriate for a charged black hole confined to a region bounded by a surface of fixed dilaton field and temperature. By making some simplifying assumptions about the quantum theory, the Hamiltonian partition function is obtained. We then use the general formalism to study the partition function for a rotating BTZ black hole confined to a box of fixed radius and temperature. [S0556-2821(99)00210-6]

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I. INTRODUCTION

The microscopic origin of black hole entropy is currently a subject of intense investigation. The Bekenstein-Hawking entropy [1] of certain extremal and near extremal black holes has been successfully derived by counting states in the large coupling limit of string theory [2]. It is important to keep in mind, however, that several other, very different, approaches have also achieved a measure of success [3–5]. For example, Carlip [3] has counted edge states in the gauge theory formulation of 2+1 gravity and obtained the correct entropy for the Banados-Teitelboim-Zanelli (BTZ) black hole [6]. This calculation has taken on new importance with the realization that many of the string inspired black holes can be related to the BTZ geometry either by looking at their near horizon geometry [7], or by using M theory inspired duality arguments [8]. This suggests that the correct explanation for black hole entropy might not necessarily be tied to a specific microscopic theory, nor to any specific low energy gravity theory: it might in some sense be universal [4]. It is therefore of interest to examine the statistical mechanics of black holes in a large variety of theories, in order to look for model independent features. A particularly useful arena for such investigations is generic dilaton gravity in two spacetime dimensions. This class of theories provides a large number of diffeomorphism invariant, solvable theories of gravity that admit black hole solutions. Moreover, there are several specific models in this class that are of direct physical significance, such as spherically symmetric gravity [9] and Jackiw-Teitelboim gravity [10]. The latter is important because its black hole solutions correspond to the dimensionally reduced BTZ black hole [11].

The study of the Hamiltonian thermodynamics of black holes in generic vacuum dilaton gravity was started in [12], generalizing a formalism first applied by Louko and Whiting [13] to spherically symmetric gravity. The purpose of the present work is to extend the results of [12] to include coupling to an Abelian gauge field. In particular we calculate the Hamiltonian partition function for a charged black hole confined to a ''box'' of fixed dilaton size. Our generic results contain as special cases all the black holes previously analyzed [14] using Louko and Whiting's formalism, and provide a unified treatment of a large variety of charged black holes. As a concrete, and interesting application, we will use our results to construct the Hamiltonian partition function for the rotating BTZ black hole, which, to the best of our knowledge, has not previously been studied.

The paper is organized as follows. In Sec. II we review generic dilaton gravity coupled to an Abelian gauge field. We present the most general solution as well as a description of the thermodynamic properties of black holes in the generic theory. For completeness, we include in the action a topological term involving the Abelian field strength. This term can only be constructed in two spacetime dimensions and has not been considered in previous work. In Sec. III, the Hamiltonian analysis of the theory is summarized, while Sec. IV derives the boundary terms that must be added to the Hamiltonian when considering a charged black hole in a box. Section V presents the Hamiltonian partition function using the results of Sec. IV and examines the resulting thermodynamics in the semi-classical, or saddle-point approximation. In Sec. VI we analyze the rotating BTZ black hole. Finally, Sec. VII summarizes our results and discusses prospects for future work.

II. GENERIC DILATON GRAVITY WITH ABELIAN GAUGE FIELD

In two spacetime dimensions, the Einstein tensor vanishes identically. In order to construct a dynamical theory of gravity with no more than two derivatives of the metric in the action, it is necessary to introduce a scalar field, traditionally called the dilaton. In the past, the dilaton was treated as essentially a Lagrange multiplier, with no physical or geometrical significance. In recent years, however, it has be-

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come clear that the dilaton plays an important role. For example, when the dilaton theory is derived via dimensional reduction by imposing spherical symmetry in (n+2)-dimensional Einstein gravity, the dilaton has a geometrical interpretation as the invariant radius of the n-sphere. More generally, the dilaton is instrumental in determining both the symmetries and the topology of the solutions [15].

In the following, we consider the most general action functional depending on the metric tensor $\bar{g}_{\mu\nu}$, scalar field $\bar{\phi}$ and Abelian gauge field in two spacetime dimensions [16,17]:

$$\bar{S}[\bar{g},\bar{\phi},A]=\int d^2x\sqrt{-\bar{g}}\Bigg[\frac{1}{2G}\left(\frac{1}{2}\bar{g}^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\bar{\phi}+\frac{1}{l^2}\bar{V}(\bar{\phi})+D(\bar{\phi})R(\bar{g})\right)-\frac{1}{4}\bar{W}(\bar{\phi})F^{\mu\nu}F_{\mu\nu}+\frac{\bar{Z}(\bar{\phi})}{\sqrt{-\bar{g}}}\epsilon^{\mu\nu}F_{\mu\nu}\Bigg], \tag{1}$$

where G is the dimensionless 2D Newton constant, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and l is a fundamental constant with dimensions of length. In addition, $\bar{V}(\bar{\phi})$, $D(\bar{\phi})$, $\bar{W}(\bar{\phi})$ and $\bar{Z}(\bar{\phi})$ are arbitrary functions of the dilaton $\bar{\phi}$. The last term in the action is a topological term that is only possible in two spacetime dimensions.[1]

It is convenient to eliminate the kinetic term for the scalar field. This can be done with an invertible field redefinition providing that $D(\bar{\phi})$ is a differentiable function of $\bar{\phi}$ such that $D(\bar{\phi})\neq 0$ and $dD(\bar{\phi})/d\bar{\phi}\neq 0$ for any admissable value of $\bar{\phi}$ [16,18]:

$$g_{\mu\nu}=\Omega^2(\bar{\phi})\bar{g}_{\mu\nu} \tag{2}$$

$$\phi=D(\bar{\phi}) \tag{3}$$

where

$$\Omega^2(\bar{\phi})=\exp\left(\frac{1}{2}\int\frac{d\bar{\phi}}{(dD/d\bar{\phi})}\right). \tag{4}$$

The electromagnetic potential is left unchanged. In terms of the new fields, the action Eq. (1) takes the form

$$S[g,A]=\frac{1}{2G}\int d^2x\sqrt{-g}\left(\phi R(g)+\frac{1}{l^2}V(\phi)\right) \tag{5}$$

$$+\int d^2x\left(-\frac{1}{4}\sqrt{-g}W(\phi)F^{\mu\nu}F_{\mu\nu}+Z(\phi)\epsilon^{\mu\nu}F_{\mu\nu}\right), \tag{6}$$

where V, $W(\phi)$ and $Z(\phi)$ are defined as

$$V(\phi)=\frac{\bar{V}(\bar{\phi}(\phi))}{\Omega^2(\bar{\phi}(\phi))} \tag{7}$$

$$W(\phi)=\Omega^2(\bar{\phi}(\phi))\bar{W}(\bar{\phi}(\phi)) \tag{8}$$

$$Z(\phi)=\bar{Z}(\bar{\phi}(\phi)). \tag{9}$$

We henceforth consider the action only in the form Eq. (6), keeping in mind that the physical metric in general may be different from $g_{\mu\nu}$.[2] The field equations are

$$R+\frac{1}{l^2}\frac{dV}{d\phi}-\frac{G}{2}\frac{dW(\phi)}{\phi}F^{\alpha\beta}F_{\alpha\beta}+\frac{2G}{\sqrt{-g}}\frac{dZ}{d\phi}\epsilon^{\alpha\beta}F_{\alpha\beta}=0 \tag{10}$$

$$\nabla_\mu\nabla_\nu\phi-\frac{1}{2l^2}g_{\mu\nu}V(\phi)-\frac{3}{4}Gg_{\mu\nu}W(\phi)F^{\alpha\beta}F_{\alpha\beta}+GW(\phi)F^{\gamma}_{\mu}F_{\nu\gamma}=0 \tag{11}$$

$$\nabla_\mu\left(W(\phi)F^{\mu\nu}-2\frac{\epsilon^{\mu\nu}}{\sqrt{-g}}Z(\phi)\right). \tag{12}$$

It follows directly from the above field equations that on shell all the fields are left invariant by Lie derivation along the following Killing vector [15]:

$$k^\mu=l\epsilon^{\mu\nu}\partial_\nu\phi/\sqrt{-g} \tag{13}$$

where $\epsilon^{\mu\nu}$ is the contravariant Levi-Civita symbol: ($\epsilon^{01}=-\epsilon^{10}=1$, etc.) and the constant l has been included to ensure that the vector components are dimensionless.

The most general solution to the field equations without the topological term has been found in [17]. The procedure required with the extra term is virtually identical, so we will omit the details. The solution for the Abelian gauge field is

$$F=\frac{1}{W(\phi)}(q-2Z(\phi)) \tag{14}$$

where q is a constant that corresponds to the Abelian charge. In the above, F is a scalar defined implicitly by $F^{\mu\nu}=FE^{\mu\nu}$, where $E^{\mu\nu}=\epsilon^{\mu\nu}/\sqrt{-g}$ is the fundamental alternating tensor. It is most convenient to write the final solutions in manifestly static coordinates by exploiting the form of the Killing vector given above. That is, we can choose the spatial coordiate to be proportional to the dilaton field:

$$\phi=x/l. \tag{15}$$

In these coordinates, the metric depends only on x:

$$ds^2=-[j(\phi)-2GlM-l^2GK(\phi;q)]dt^2+[j(\phi)-2GlM-l^2GK(\phi;q)]^{-1}dx^2 \tag{16}$$

[1]G.K. is grateful to R. Jackiw for pointing out this possibility.

[2]It is crucial in this regard that the black hole thermodynamics are invariant under conformal reparametrizations of the form Eq. (2).

where M is a constant of integration, which will be shown below to be the Arnowitt-Deser-Misner (ADM) mass of the solution and we have defined

$$j(\phi)=\int_0^\phi d\tilde{\phi} V(\tilde{\phi}) \tag{17}$$

$$K(\phi;q)=\int_0^\phi d\tilde{\phi}(q-2Z(\tilde{\phi}))^2/W(\tilde{\phi}). \tag{18}$$

The general solution has an apparent horizon at the surface $\phi=\phi_0=\text{const}$ for ϕ_0 given by

$$f(\phi_0)=0 \tag{19}$$

where we have defined

$$f(\phi;M,q):=[j(\phi)-2GlM-l^2GK(\phi;q)]. \tag{20}$$

The global form of the solution, and in particular the number of horizons, depends on the specific forms of the function $j(\phi)$ and $K(\phi;q)$.

We now review the thermodynamic properties of the solutions. Specifically, we assume that ϕ_0 is the value of the dilaton field at an exterior, bifurcative horizon. A straightforward calculation reveals that the surface gravity at the horizon is

$$\kappa:=\sqrt{-\frac{1}{2}\nabla^\mu k^\nu \nabla_\mu k_\nu}\Bigg|_{\phi_0} =\frac{f'(\phi_0)}{2l}=\frac{V(\phi_0)}{2l}-\frac{l(q-2Z(\phi_0))^2G}{2W(\phi_0)} \tag{21}$$

where the prime denotes differentiation with respect to ϕ.

The Hawking temperature of the horizon can be calculated by analytically continuing the solution exterior to the horizon to Euclidean time, imposing periodicity in the imaginary time direction and requiring the resulting solution to be regular at the horizon. The resulting Hawking temperature is

$$T_H=\frac{f'(\phi_0;M,q)}{4\pi l}. \tag{22}$$

As discussed in [15], the expression for the black hole entropy can most easily be derived by demanding that the first law of thermodynamics be satisfied with respect to infinitesimal variations of the mass and charge of the black hole. In particular, if we vary the parameters M and q of the solution while staying on the event horizon, $f=0$, we get the condition on the corresponding variation of ϕ_0 at the horizon:

$$\begin{aligned}0&=\frac{\partial f}{\partial \phi_0}\delta\phi_0+\frac{\partial f}{\partial M}\delta M+\frac{\partial f}{\partial q}\delta q\\ &=\left(V(\phi_0)-\frac{l^2G(q-2Z(\phi_0))^2}{W(\phi_0)}\right)\delta\phi_0-2Gl^2\delta M\\ &\quad-\mathcal{P}(\phi_0,q)\delta q\end{aligned} \tag{23}$$

where

$$\mathcal{P}(\phi_0,q)=\int^{\phi_0} d\phi\frac{(q-2Z(\phi))}{W(\phi)}. \tag{24}$$

This yields the first law of black hole thermodynamics,

$$\delta M=T_H\delta S_{BH}-\mathcal{P}\delta q, \tag{25}$$

where we have defined the Bekenstein-Hawking entropy:

$$S_{BH}(M,q)=\frac{2\pi}{G}\phi_0(M,q) \tag{26}$$

where $\phi_0(M,q)$ is obtained by solving Eq. (19). Equation (25) also shows that $\mathcal{P}$ is the generalized force associated with the charge q.

III. HAMILTONIAN ANALYSIS

The Hamiltonian analysis for generic dilaton gravity has been presented in many works. Here we summarize the results, using the notation and conventions of [17]. We start by decomposing the metric as follows:

$$ds^2=e^{2\rho}[-u^2dt^2+(dx+v\,dt)^2], \tag{27}$$

where x is a local coordinate for the spatial section Σ and ρ, u and v are functions of spacetime coordinates (x,t). In terms of this parametrization, we find the canonical Hamiltonian (up to surface terms which will be discussed below):

$$H_c=\int dx\left(v\mathcal{F}+\frac{u}{2G}\mathcal{G}+A_0\mathcal{J}\right) \tag{28}$$

where Π_ϕ, Π_ρ and Π_{A_1} are the momenta conjugate to ϕ, ρ and A_1, respectively while

$$\mathcal{F}=\rho'\Pi_\rho+\phi'\Pi_\phi-\Pi_\rho'\sim 0 \tag{29}$$

$$\mathcal{G}=2\phi''-2\phi'\rho'-2G^2\Pi_\phi\Pi_\rho-e^{2\rho}\frac{V(\phi)}{l^2} +\frac{Ge^{2\rho}}{W(\phi)}[\Pi_{A_1}-2Z(\phi)]^2\sim 0 \tag{30}$$

$$\mathcal{J}=-\Pi_{A_1}' \tag{31}$$

are secondary constraints. Note that $\mathcal{F}$ annd $\mathcal{G}$ generate spatial and temporal diffeomorphisms, while $\mathcal{J}$ is the Gauss law constraint that generates Abelian gauge transformations.

The general solution presented in the previous section suggests that there are two independent, diffeomorphism invariant physical observables, namely the mass of the black hole and its Abelian charge. These observables can easily be expressed in terms of the phase space variables. In particular, define

$$\mathcal{Q}=\Pi_{A_1}. \tag{32}$$

The constant mode q of $\mathcal{Q}$ is therefore a physical observable and corresponds precisely to the Abelian charge in the solution Eq. (14). Similarly, we can define the mass observable:

$$\mathcal{M}=\frac{l}{2G}\left(e^{-2\rho}(G^2\Pi_\rho^2-(\phi')^2)+\frac{j(\phi)}{l}-GK(\phi,\mathcal{Q})\right) \tag{33}$$

where

$$K(\phi,\mathcal{Q}):=\int^{\phi}d\tilde{\phi}\frac{(\mathcal{Q}-2Z(\tilde{\phi}))^2}{W(\phi)}. \tag{34}$$

$\mathcal{M}$ commutes with the constraints and is spatially constant on the constraint surface since

$$\frac{\partial\mathcal{M}}{\partial x}=-le^{-2\rho}\left(G\Pi_\rho\mathcal{F}+\frac{1}{2G}\phi'\mathcal{G}-e^{2\rho}\mathcal{P}(\phi,\mathcal{Q})\mathcal{J}\right) \tag{35}$$

where

$$\mathcal{P}(\phi,\mathcal{Q})=\int d\phi\frac{(\Pi_{A_1}-2Z(\phi))}{W(\phi)}. \tag{36}$$

The constant mode of $\mathcal{M}$ is the mass parameter appearing in the solution Eq. (16). Although the observables $\mathcal{M}$ and $\mathcal{Q}$ are invariant under general diffeomorphisms, their conjugates $\Pi_{\mathcal{M}}$ and $\Pi_{\mathcal{Q}}$ are only invariant with respect to diffeomorphisms that vanish on the boundaries of the system [17].

IV. BOUNDARY TERMS IN THE HAMILTONIAN

The previous section neglected the boundary terms that must be added to the canonical Hamiltonian in order that the variational principle be well defined. These depend on the boundary conditions and define the canonical energy, since the remainder of the Hamiltonian vanishes on the constraint surface. We now derive the boundary terms for boundary conditions corresponding to a charged black hole in a box of fixed, constant ‘‘radius’’ (surface of constant dilaton field). For convenience we rewrite the canonical Hamiltonian as follows:

$$H_c=\int_{\sigma_-}^{\sigma_+}dx(\tilde{u}\tilde{\mathcal{G}}+\tilde{v}\mathcal{F}+\tilde{A}\mathcal{J})+H_+-H_- \tag{37}$$

where have replaced the original Hamiltonian constraint $\mathcal{G}$ by the linear combination of constraints corresponding to the spatial derivative of the mass observable:

$$\tilde{\mathcal{G}}=-\frac{\partial\mathcal{M}}{\partial x}=le^{-2\rho}\left(G\Pi_\rho\mathcal{F}+\frac{1}{2G}\phi'\mathcal{G}-e^{2\rho}\mathcal{P}\mathcal{J}\right) \tag{38}$$

and replace the original Lagrange multipliers by

$$\tilde{u}=\frac{ue^{2\rho}}{l\phi'} \tag{39}$$

$$\tilde{v}=v-\frac{uG\Pi_\rho}{\phi'} \tag{40}$$

$$\tilde{A}=A_0+\frac{ue^{2\rho}}{\phi'}\mathcal{P}. \tag{41}$$

H_+ and H_- are previously neglected boundary terms determined by the requirement that the surface terms in the variation of H_c vanish for a given set of boundary conditions.

We wish to consider the (1+1)-dimensional analogue of a charged black hole in a box of fixed radius. We will therefore keep the value of the dilaton at the outer boundary $\phi_+:=\phi(\sigma_+)$ fixed and independent of time, as well as the component of the metric along the world line of the box $[g_{tt}^+:=g_{tt}(\sigma_+)]$. The relevant boundary conditions on the vector potential are $A_1(\sigma_+)=0$ and $A_0(\sigma_+)=A_0^+=\text{const}$. Give the above conditions, the boundary variation of the canonical Hamiltonian Eq. (37) at σ_+ will vanish if

$$\delta H_+(\mathcal{M},\mathcal{Q})=\tilde{u}\,\delta\mathcal{M}|_{\sigma_+}+\tilde{a}\,\delta\mathcal{Q}|_{\sigma_+}. \tag{42}$$

Since

$$\tilde{u}_+=\left(\frac{g_{tt}^+}{2G\mathcal{M}l-j(\phi_+)+l^2GK(\phi_+,\mathcal{Q})}\right)^{1/2} \tag{43}$$

$$\tilde{A}_+=A_0^++\frac{l}{2}\tilde{u}(\sigma_+)\frac{\partial K(\phi_+,\mathcal{Q})}{\partial\mathcal{Q}}\bigg|_{\sigma_+}. \tag{44}$$

Equation (42) can be directly integrated to yield

$$H_+(\mathcal{M},\mathcal{Q})=\frac{\sqrt{-g_{tt}^+j(\phi_+)}}{lG}\times\left(1-\sqrt{1-\frac{2Gl\mathcal{M}}{j(\phi_+)}-\frac{l^2GK(\phi_+,\mathcal{Q})}{j(\phi_+)}}\right)+A_0^+\mathcal{Q}. \tag{45}$$

Note that we have chosen the constant of integration so as to guarantee that $H_+=0$ when $\mathcal{M}=\mathcal{Q}=0$. If $K(\phi_+,\mathcal{Q})$ remains finite as $\phi_+\to\infty$, then

$$H_+(\mathcal{M},\mathcal{Q})\to\sqrt{\frac{-g_{tt}^+}{j(\phi_+)}}\mathcal{M}. \tag{46}$$

Hence, on the constraint surface, $\mathcal{M}$ is proportional to the ADM mass. The value of the constant of proportionality will depend on the boundary conditions on the metric and ϕ_+. This will be discussed in more detail below.

We next consider the inner boundary σ_-. Following the work of Louko and Whiting [13] we require our spatial slices to approach the bifurcation point ($k^\mu=0$) of the black hole along a static slice. These boundary conditions are natural for the consideration of the thermodynamics of the black hole, since the resulting spacetimes can be analytically continued to the Euclidean spacetime described by the nonsingular Gibbons-Hawking instanton. Given the general form of the Killing vector in Eq. (13), for a static slice ($\dot{\phi}_-=0$), the condition that σ_- be a bifurcation point reduces to

$$\phi'(\sigma_-)=0. \tag{47}$$

From the thermodynamic considerations of Sec. II, it follows that the metric on the inner boundary must approach the form

$$ds^2 \to -R^2(dt/\tilde{a})^2+H(R)dR^2 \tag{48}$$

where $R=0$ at the bifurcation point σ_-, $H(0)=1$ and $2\pi\tilde{a}$ equals the periodicity of the Euclidean time required to make the Euclidean solution regular at the horizon.[3] The required boundary conditions on the metric components in (t,R) coordinates are therefore

$$e^{2\rho(\sigma_-)}=1 \tag{49}$$

$$v(\sigma_-)=0 \tag{50}$$

$$u(\sigma_-)=0 \tag{51}$$

$$u'(\sigma_-)=\frac{1}{\tilde{a}}. \tag{52}$$

Since, in terms of phase space coordinates,

$$|k|^2=l^2e^{-2\rho}((G\pi_\rho)^2-\phi'^2) \tag{53}$$

we must also impose the condition

$$\pi_\rho(\sigma_-)=0 \tag{54}$$

to ensure that $|k|^2_-=0$.

Finally, following Louko and Winters-Hilt [14], we choose the boundary conditions on the U(1) vector potential at the bifurcation point to be

$$A_1(\sigma_-)=0 \tag{55}$$

$$A_0(\sigma_-)=A_0^-=\text{const.} \tag{56}$$

With the above boundary conditions we find

$$\tilde{v}(\sigma_-)=0 \tag{57}$$

$$\tilde{u}(\sigma_-)=\frac{2l}{\tilde{a}\tilde{V}(\phi_-,\mathcal{Q})} \tag{58}$$

$$\tilde{A}(\sigma_-)=\frac{l^2}{\tilde{a}\tilde{V}(\phi_-,\mathcal{Q})}\frac{\partial K(\phi_-,\mathcal{Q})}{\partial \mathcal{Q}}+A_0^- \tag{59}$$

where we have defined

$$\tilde{V}(\phi,\mathcal{Q})=V(\phi)-Gl^2\frac{\partial K(\phi,\mathcal{Q})}{\partial \phi}. \tag{60}$$

With these boundary conditions there will be no boundary terms at σ_- from the variation of the Hamiltonian if

$$\delta H_-=\frac{2l}{\tilde{a}\tilde{V}(\phi_-,\mathcal{Q})}\delta\mathcal{M}\bigg|_{\sigma_-}+\frac{l^2}{\tilde{a}\tilde{V}(\phi_-,\mathcal{Q})}\frac{\partial K(\phi_-,\mathcal{Q})}{\partial \mathcal{Q}}\bigg|_{\sigma_-}\delta\mathcal{Q}+A_0^-\delta\mathcal{Q}. \tag{61}$$

Next we use the fact that the norm of the Killing vector is constrained to vanish at the inner boundary to obtain

$$\delta\mathcal{M}=\frac{1}{2Gl}\tilde{V}(\phi_-,\mathcal{Q})\delta\phi_- -\frac{l}{2}\frac{\partial K(\phi_-,\mathcal{Q})}{\partial \mathcal{Q}}\delta\mathcal{Q}. \tag{62}$$

Substituting this into Eq. (61) and simplifying gives

$$\delta H_-=\frac{1}{\tilde{a}G}\delta\phi_- +A_0^-\delta\mathcal{Q} \tag{63}$$

which can be trivially integrated to yield

$$H_-(\mathcal{M},\mathcal{Q})=\frac{1}{\tilde{a}G}\phi_-(\mathcal{M},\mathcal{Q})+A_0^-\mathcal{Q}. \tag{64}$$

By using Eq. (26) our final expression for the canonical Hamiltonian on the constraint surface takes the simple form

$$H_c=E(M,q;\phi_+)-\frac{1}{2\pi\tilde{a}}S_{B.H.}(M,q)-\gamma q \tag{65}$$

where

$$E(M,q;\phi_+)=\frac{\sqrt{-g_{tt}^+ j(\phi_+)}}{Gl}\times\left(1-\sqrt{1-\frac{2GMl}{j(\phi_+)}-\frac{l^2GK(\phi_+,q)}{j(\phi_+)}}\right) \tag{66}$$

[3]Recall that the time coordinate in this section is normalized so that g_{tt}^+ is fixed. The parameter $\tilde{a}$ therefore differs from a in Sec. II.

is the quasilocal energy and $\gamma \equiv A_0^- - A_0^+$. We have also used the fact that on the constraint surface $\mathcal{M}=M$ and $\mathcal{Q}=q$, where M and q correspond to the physical mass and charge appearing in the general solution Eq. (16).

In addition to the dynamical variables M and q, the canonical Hamiltonian appears to depend on four fixed external parameters, g_{tt}^+, ϕ_+, $\tilde{a}$ and γ. ϕ_+ plays the role of the effective box size, while γ is analogous to a chemical potential. g_{tt}^+ and $\tilde{a}$ on the other hand must be fixed by imposing further boundary conditions. In particular, the metric g_{tt}^+ is related to the choice of time coordinate along the boundary. This is normally chosen to equal the proper time as measured with respect to a given physical metric. In vacuum dilaton gravity, the choice of physical metric is subtle since one can always do conformal reparametrizations involving the dilaton. One must therefore define the ''physical metric'' to be the one which determines the geodesics of massive test particles. For now we will consider the most general case and write

$$g_{\mu\nu} = h(\phi) g_{\mu\nu}^{phys} \tag{67}$$

where $h(\phi)$ is an arbitrary function of ϕ that must ultimately be determined experimentally. If $g_{tt}^{phys}(\sigma_+) = -1$, then

$$g_{tt}^+ = -h(\phi_+). \tag{68}$$

The constant $\tilde{a}$ must be fixed by thermodynamic considerations [13]. We have already shown that $2\pi\tilde{a}$ must be equal to the period of the corresponding Euclidean time in order for the Euclideanized solution to be regular at the horizon. In the Euclidean formulation of black hole thermodynamics, the inverse temperature β at the boundary of the system is

$$\beta = \sqrt{-g_{tt}^{phys}(\sigma_+)}\, 2\pi\tilde{a} = 2\pi\tilde{a}. \tag{69}$$

The final form of the canonical Hamiltonian is therefore

$$H_c = E(M,q,\phi_+) - \beta^{-1} S_{B.H.}(M,q) - \gamma q \tag{70}$$

where

$$E(M,q;\phi_+) = \frac{\sqrt{h(\phi_+) j(\phi_+)}}{Gl} \times \left(1 - \sqrt{1 - \frac{2GMl}{j(\phi_+)} - \frac{l^2 G^2 K(\phi_+,q)}{j(\phi_+)}}\right). \tag{71}$$

V. HAMILTONIAN PARTITION FUNCTION

The quantum partition function of interest is formally defined as

$$Z[\beta,\phi_+,\gamma] = \mathrm{Tr}[\exp(-\beta\hat{H})] \tag{72}$$

where the trace is over all physical states and β corresponds to the (fixed) temperature at the boundary of the system. This trace is most easily expressed in term of the eigenstates $|M,Q\rangle$ of the mass and charge operators:

$$Z(\beta,\phi_+,\gamma) = \int dM \int dQ\, \mu(M,Q) \langle M,Q| e^{-\beta\hat{H}} |M,Q\rangle. \tag{73}$$

In the above, $\mu(M,Q)$ is an as yet unknown measure on the space of observables. Following Louko and Whiting [13], we will make the simplest, physically reasonable assumptions about the measure and the allowed values of M and Q. A more rigorous derivation of the measure will be addressed in future work. First of all, we restrict the ADM mass M to be positive. Secondly, we allow only those value of M and Q for which at least one bifurcative horizon exists where $f(\phi)$ has a simple zero (i.e., no extremal black holes or naked singularities). Finally, we require the value of the dilaton at the horizon to be less than its value at the boundary of the system (i.e., the box must lie outside the horizon) so that equilibrium is in fact possible. With these assumptions the space of allowed values for the observables is finite. This will be made explicit for specific examples in the next section.

As in [13] (see also [19]) we assume that

$$\mu(M,Q)\langle M,Q|M,Q\rangle = \frac{1}{\mathcal{V}} \tag{74}$$

where $\mathcal{V}$ is the volume of the allowed space of observables. The final expression for the partition function is therefore

$$Z(\beta,\phi_+,\gamma) = \mathcal{V}^{-1} \int_{\mathcal{V}} dM\, dq\, e^{S_{BH}(M,q)} e^{-\beta(E(M,q,\phi_+) - \gamma q)}. \tag{75}$$

Note that the Bekenstein-Hawking entropy enters the partition function as the logarithm of an apparent degeneracy of the physical mass and charge eigenstates. Moreover, q is thermodynamically analoguous to particle number, while γ plays the role of a chemical potential.

The above expression, can in principle be integrated to yield the partition function describing the thermodynamics charged black holes in a box for any particular dilaton gravity theory. We will now show that it gives the correct classical thermodynamic behavior in the saddle-point approximation. In this approximation, the choice of measure is irrelevant except in the unlikely event that it is exponential in the observables. Thus, we have

$$Z(\beta,\phi_+,\gamma) \approx e^{-I(\bar{M},\bar{q},\beta,\phi_+,\gamma)} \tag{76}$$

where we have defined

$$I(M,q,\beta,\phi_+,\gamma) = \beta(E(M,q,\phi_+) - \gamma q) - S_{BH}(M,q) \tag{77}$$

and $\bar{M}$ and $\bar{q}$ are the values of the mass and charge at the minimum of I (if one exists). The equation obtained by extremizing with respect to M is

$$\beta=\sqrt{\frac{f(\phi_+,\bar{M},\bar{q})}{h(\phi_+)}}\beta_H(\bar{M},\bar{q}) \tag{78}$$

which implies that the temperature at the boundary is equal to the red-shifted Hawking temperature $\beta_H=1/T_H=4\pi l/f'(\phi_-,\bar{M},\bar{q})$ associated with the mean mass and charge.

Variation with respect to q gives for the chemical potential:

$$\gamma=\frac{l}{2}\frac{\beta_H(\bar{M},\bar{q})}{\beta}\left(\frac{\partial K(\phi_+,q)}{\partial q}-\frac{\partial K(\phi_-,q)}{\partial q}\right)\bigg|_{\bar{M},\bar{q}} \tag{79}$$

where as $\phi_-=\phi_-(M,q)$ as determined by Eq. (19). Using Eq. (76) we can evaluate the mean energy, mean charge and entropy of the system:

$$\langle E\rangle=-\frac{\partial \ln(Z)}{\beta}\bigg|_{\bar{M},\bar{q}}+\frac{\gamma}{\beta}\frac{\partial \ln(Z)}{\partial\gamma}\bigg|_{\bar{M},\bar{q}} \approx E(\bar{M},\bar{q},\phi_+) \tag{80}$$

$$\langle q\rangle=\beta^{-1}\left|\frac{\partial \ln(Z)}{\partial\gamma}\right|_{\bar{M},\bar{q}}\approx\bar{q} \tag{81}$$

$$S=\left(1-\beta\frac{\partial}{\partial\beta}\right)\ln(Z)=S_{BH}(\bar{M},\bar{q}). \tag{82}$$

A straightforward calculation verifies that the above expressions for the mean energy, charge and entropy automatically obey the generalized first law

$$\delta\langle E\rangle=\frac{\partial E}{\partial M}\delta\bar{M}+\frac{\partial E}{\partial q}\delta\bar{q}+\frac{\partial E}{\partial\phi_+}\delta\phi_+ =\beta^{-1}\delta S_{BH}+\gamma\delta\langle q\rangle-\mathcal{W}\delta\phi_+ \tag{83}$$

where

$$\mathcal{W}:=-\frac{\partial E(M,q,\phi_+)}{\partial\phi_+}\bigg|_{\bar{M},\bar{q}} \tag{84}$$

is a generalized surface pressure: it is the rate of change of quasilocal energy with ‘‘box size.’’

It can be verified that in the case of spherically symmetric gravity, the above generic expressions for the mean energy, entropy, etc. correctly reproduce earlier results [20] for the semi-classical thermodynamics of Reissner-Nordstrom black holes. We now consider a specific and interesting case that has not been analyzed in previous work: the rotating BTZ black hole.

VI. THE ROTATING BTZ BLACK HOLE

Starting with the Einstein action with cosmological constant in 2+1 dimensions:

$$I^{(3)}=\frac{1}{16\pi G^{(3)}}\int d^3x\sqrt{-g^{(3)}}(R(g^{(3)})+\Lambda). \tag{85}$$

In 2+1 dimensions, the gravitational constant $G^{(3)}$ has dimensions of length. We now impose axial symmetry by considering metrics of the form[4]

$$ds^2_{(3)}=g_{\mu\nu}dx^\mu dx^\nu+\phi(x)^2(a\,d\theta+A_\mu dx^\mu)^2 \tag{86}$$

where a is an arbitrary constant with dimensions of length which, without loss of generality we take to be proportional to the (2+1)-dimensional Planck length $a=8G^{(3)}$. The one-form components A_μ are dimensionless. Unless the one-form $A=A_\mu dx^\mu$ is closed, the metric is not static so that the field strength $F_{\mu\nu}=A_{\mu,\nu}-A_{\nu,\mu}$ is proportional to the angular momentum of the solution. With the above metric ansatz the reduced action is that of Jackiw-Teitelboim dilaton gravity coupled to an Abelian gauge field:

$$I^{(2)}=\int d^2x\sqrt{-g}\left(\phi R(g)+\phi\Lambda-\frac{1}{4}\phi^3F^{\mu\nu}F_{\mu\nu}\right). \tag{87}$$

This action is already of the generic form Eq. (6) without the need for further definitions. In particular, $G=1/2$, $l=\Lambda^{-1/2}$, $V(\phi)=\phi$ and $W(\phi)=\phi^3$. Choosing $r=l\phi$ as the spatial coordinate the general solution takes the form

$$ds^2=-f(r,M,J)dt^2+\frac{1}{f(r,M,J)}dr^2 \tag{88}$$

where

$$f(r,M,J)=\left(\frac{r^2}{2l^2}-Ml+\frac{J^2l^4}{4r^2}\right). \tag{89}$$

As mentioned above, the Abelian charge J in this case is the angular momentum of the black hole. For non-zero J there are again two event horizons, at

$$r_{o,i}=l(Ml\pm\sqrt{(Ml)^2-(Jl)^2/2})^{1/2} \tag{90}$$

where r_o (r_i) is the outer (inner) horizon. The associated entropy is

$$S=4\pi\frac{r_o}{l}=4\pi\phi(r_o)=\frac{A}{4G^{(3)}} \tag{91}$$

where $A=2\pi a\phi(r_o)=16\pi G^{(3)}\phi(r_o)$ is the invariant circumference of the outer horizon, as calculated from Eq. (86). The Bekenstein Hawking entropy can also be calculated directly from Eq. (22) to be

$$T_{BH}=\frac{1}{4\pi l^2}\left(\frac{r_o^2-r_i^2}{r_o}\right). \tag{92}$$

[4]In 2+1 dimensions, there is a generalized Birkhoff theorem which states that all solutions have axial symmetry, and are stationary.

In the semi-classical approximation, the mean energy of a black hole in a box of fixed temperature and radius is

$$\langle E\rangle=\frac{\sqrt{2}r_+}{l^2}\left(1-\sqrt{1-\frac{2\bar{M}l^3}{r_+^2}+\frac{\bar{J}^2l^6}{2r_+^4}}\right) \tag{93}$$

where $\bar{M}$ and $\bar{J}$ are the mean mass and angular momentum. Note that we have used the fact that the physical metric is $g_{\mu\nu}$ in this case, so that $h(\phi_+)=1$. The physical metric is not asymptotically flat (it is in fact a metric of constant curvature) which accounts for the strange asymptotic behavior of the mean energy as the box size goes to infinity. One can invert this relation to express the mass in terms of the mean energy:

$$\bar{M}=\langle E\rangle-\frac{\langle E\rangle^2l^3}{2r_+^2}+\frac{\langle J\rangle^2l^3}{4r_+^2}. \tag{94}$$

It is also straightforward to calculate the chemical potential. It is

$$\gamma=-\frac{Jl^3\sqrt{1-r_o^2/r_+^2}}{2r_o^2\sqrt{1-\frac{J^2l^6}{2r_o^2r_+^2}}} \tag{95}$$

which approaches

$$\gamma\to-\frac{Jl^3}{2r_o^2} \tag{96}$$

as $r_+\to\infty$.

Finally, we calculate the allowed volume $\mathcal{V}$ of the physical configuration space. We wish to restrict the values of M and J so that there is always at least one positive, nondegenerate root for $f(r,M,J)=0$. This requires $M>0$ and from Eq. (90) $M>J/\sqrt{2}$. For the box size to be greater than the radius of the outer horizon, we also require

$$M<\frac{J^2l^3}{4r_+^2}+\frac{r_+^2}{2l^3}. \tag{97}$$

The volume of the allowed observable space is therefore

$$\mathcal{V}=\int_{-\sqrt{2}r_+^2/l^3}^{\sqrt{2}r_+^2/l^3}dJ\int_{J/\sqrt{2}}^{J^2l^3/4r_+^2+r_+^2/2l^3}dM=\frac{\sqrt{2}}{3}\frac{r_+^4}{l^6}. \tag{98}$$

VII. CONCLUSIONS

We have calculated the Hamiltonian partition function for generic dilaton gravity coupled to an Abelian gauge field. The class of theories considered contains many specific charged black holes of physical interest. For example, our formalism gives the correct partition function in the saddle point approximation for spherically symmetric gravity. The generic results were used to obtain the partition function for a rotating BTZ black hole confined to a box of fixed radius and temperature.

In principle the partition function that we derived can be integrated exactly. In practice, however, a numerical analysis is required in order to go beyond the semi-classical approximation. In a subsequent paper, we will do such a numerical analysis for specific theories, such as the BTZ black hole, in order to gain further information about phase structure, specific heats, etc. The ansatz that we used is, however, only rigorous in the semi-classical approximation. In particular, the integration measure, although motivated by plausibility arguments, was not derived from the fundamental quantum theory, so it is likely that there are further quantum corrections that we have not been able to incorporate. A detailed analysis of the possible quantum corrections is currently in progress.

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